

Analysis of rectangular thin plates by using finite difference method

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ABSTRACT: This paper presents an investigation into the performance evaluation of Finite Difference (FD) method in modeling a rectangular thin plate structure. In case of complex and big construction systems subjected to the arbitrary loads, including a complex boundary conditions, solving of differential equations by analytical methods is almost impossible. Then the solution is application of numerical methods. The differential equations are discretized by means of the finite difference method which are used to determine the in-plane stress functions of plates and reduced to several sets of linear algebraic simultaneous equations. In the end, A problems is solved which illustrate the potential of the method for predicting the finite stress, deflection and farther directions of investigations are given. Finally, it was found that the finite difference method selection is desired to model thin plates structure.

Keywords: Finite Difference Method, Failure Thoeries, Thin Plate, Distortion Energy Theory, Strain.

INTRODUCTION

Thin plates are structural elements that their thickness is smaller than its two dimensions. Among practical examples to describe the dimensions of these plates are roof, building windows, flat part of a table, manhole thin covering and panels. plates are divided into two categories: thin plates with large deflections and thick plates (Boot, 1988 and Krysko, 2011). In thin plates, deflections and deformation of structural elements are usually considered and for ease of work, in structural plates, their deflections are studied under loading conditions.

Linear theory is used in describing the lateral displacements or small deformation under loading and this method is used in the analysis of thin plates under lateral loads. If plates' deflections are large, this method cannot be used (Kaiser, 1936).

If plates' deflections are large, deflection of middle of the plate is increased then, the linear method cannot be applied for determining the deflection of these plates. These errors are large and obtained linear solutions including displacement and stress are contrary to experimental observations.

Thus the non-linear theory for plates is developed by Von Karman and was used in the analysis of thin plates. Linear difference equations in plates were presented for the first time in 1910 by Von Karman then Kirchhoff used these equations for large deformations.

He also studied plates' internal forces with external deformations and stated simultaneously their relationship with each other. The simplest application of this method is in thin rectangular plates. Then In 1936, Kaser solved a uniformly laterally loaded, simply supported, square plate problem (Kan, 1967 and Kim, 2002).

He used finite difference method and supported solutions to solve this problem and analyzing experimental results in analyzing this plate. In recent years, studies were done in connection with finite element of flexure problems such as analysis of large displacements, plate vibration, problems related to stress, etc (Wang and Wu , 2011; Zhang, 2010).

An approximate method for the analysis of plates using the finite difference method were presented by Bhaumik and Hanley for uniformly loaded rectangular plate. However, in this study they assumed that the behavior of each point of the mesh in the thickness is fully elastic or fully plastic.

This hypothesis is to facilitate the review of plate bending. Although for some structural materials the relationship between anchor and bending using two lines is not correct (Mochneck, 2010). According what is

mentioned, enough studies have not been done in examining the application of finite difference method in modeling and simulating thin plates, Therefore, in this study, the finite difference method was evaluated using a computer program for the analysis of stress and deformation of rectangular thin plate under supported solutions and specified supports.

MATERIALS AND METHODS

Methodologies using FDM

A number of analytical approaches were proposed by different researchers to solve the plate differential equation of motion. FE (Finite Element) and FD methods are known to be the most widely used numerical procedures to solve the mentioned differential equation. An advantageous of FE method is that it is very suitable for practical engineering problems of complex geometries. However, the computational complexity involved in this method constitutes the main disadvantage of this technique, especially in realtime application.

On the other hand, the FD method is relatively easy to program, fast enough to analyze and also seems to be more convenient for uniform structures such as plate system. The main serious drawback of FD method is that it is not suitable for problems with awkward and irregular geometries (Wu et al, 2010).

Furthermore, since it is difficult to vary the size of the difference cell in particular regions, it is not suitable for problems with rapidly changing variables such as stress concentration problems. In any case, because of the geometry uniformity of the thin plates, FD method seems to be more applicable and faster to calculate especially for the case of realtime design of an active vibration controller.

In FD method, the entire solution domain is divided into a grid of cells. Then, the derivatives in the governing partial differential equations are written in terms of difference equations. Therefore, the FD is applied to each interior point so that the displacement of each node is related to the values at the other nodes in the grid connected to it. Considering the boundary conditions of the problem, a unique solution can be obtained for the overall system (Chakraborty, 2011).

Initial equations

We know from the equations of elasticity theory:

$$\begin{aligned}\sigma_x &= 2G\epsilon_x + \lambda e \\ \sigma_y &= 2G\epsilon_y + \lambda e \\ \tau_{xy} &= Gy_{xy}\end{aligned}$$

since

$$\begin{aligned}G &= \frac{E}{2(1 + \nu)} \\ \lambda &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)}\end{aligned}$$

$$e = \epsilon_x + \epsilon_y$$

Thus, Equations can be re-written as:

$$\epsilon_x = -z \frac{\partial^2 w}{\partial x^2}$$

$$\epsilon_y = -z \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}$$

Hence

$$\sigma_x = 2 \frac{E}{2(1 + \nu)} \left(-z \frac{\partial^2 w}{\partial x^2} \right) + \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left(-z \frac{\partial^2 w}{\partial x^2} - z \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_x = -\frac{Ez}{(1 - \nu^2)} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y = 2 \frac{E}{2(1 + \nu)} \left(-z \frac{\partial^2 w}{\partial y^2} \right) + \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \left(-z \frac{\partial^2 w}{\partial x^2} - z \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y = -\frac{Ez}{(1 - \nu^2)} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$$

$$\tau_{xy} = \frac{E}{2(1 + \nu)} \left(-2z \frac{\partial^2 w}{\partial x \partial y} \right)$$

$$\tau_{xy} = -\frac{Ez}{(1 + \nu)} \left(\frac{\partial^2 w}{\partial x \partial y} \right)$$

Flexural rigidity (D) is defined as

$$D = \frac{Et^3}{12(1 - \nu^2)}$$

Failure Theories

If all structures were loaded in only one direction, it would be easy to predict failure. All that would be needed was a single uniaxial test to find the yield stress and ultimate stress levels. If it is a brittle material, then the ultimate stress will determine failure. For ductile material, failure is assumed to be when the material starts to yield and permanently deform.

However, when a structure has multiple stresses at a given local (σ_x , σ_y and τ_{xy} for 2D as discussed in Stresses at a Point section), then the interaction between those stresses may effect the final failure. This section presents distortion energy that can be used for different types of materials to help predict failure when multiple stresses are applied.

For simplification, all failure theories are based on principal stresses (σ_1 , σ_2) which can be determined from any (σ_x , σ_y and τ_{xy}) stress state. This removes the shear stress terms since the shear stress is zero at the principal directions. Using principal stresses does not change the results from the failure theories (Zhang, 2010).

Maximum distortion energy theory

The maximum distortion energy theory, also known as the *von Mises theory*, was proposed by M. T. Huber in 1904 and further developed by R. von Mises (1913) and H. Hencky (1925). In this theory, failure by yielding occurs when, at any point in the body, the distortion energy per unit volume in a state of combined stress becomes equal to that associated with yielding in a simple tension test.

The distortion energy theory says that failure occurs due to distortion of a part, not due to volumetric changes in the part (distortion causes shearing, but volumetric changes due not).

This theory looks at the total energy at failure and compares that with the total energy in a uniaxial test at failure. Any elastic member under load acts like a spring and stores energy (Vallabhan, 1983). This is commonly called distortional energy and can be calculated as:

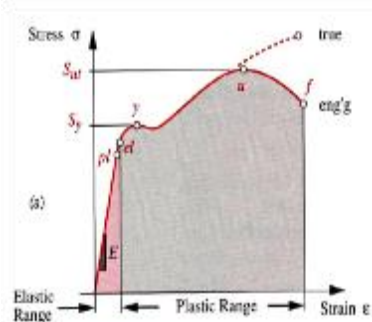


Figure.1. The area under the curve in the elastic region is called the Elastic Strain Energy

$$U = \frac{1}{2} \sigma \epsilon$$

In 3D case we have

$$U_T = \frac{1}{2} \sigma_1 \epsilon_1 + \frac{1}{2} \sigma_2 \epsilon_2 + \frac{1}{2} \sigma_3 \epsilon_3$$

Stress-strain relationship

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} - \nu \frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \nu \frac{\sigma_1}{E} - \nu \frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \nu \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E}$$

Define

Distortion strain energy = total strain energy – hydrostatic strain energy

$$U_d = U_T - U_h$$

$$U_T = \frac{1}{2E} [(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2\nu(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)]$$

Substitute $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_h$

where

$$U_h = \frac{1}{2E} [(\sigma_h^2 + \sigma_h^2 + \sigma_h^2) - 2\nu(\sigma_h\sigma_h + \sigma_h\sigma_h + \sigma_h\sigma_h)]$$

Simplify and substitute $\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h$ into the above equation

$$\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma_h$$

Then, Equation become

$$U_h = \frac{3\sigma_h^2}{2E} (1 - 2\nu) = \frac{(\sigma_1 + \sigma_2 + \sigma_3)^2 (1 - 2\nu)}{6E}$$

Subtract the hydrostatic strain energy from the total energy to obtain the distortion energy

$$U_d = U_T - U_h = \frac{(1 + \nu)}{6E} [(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

Numerical implementation

In this paper, finite difference method (FDM) was used to obtain solutions for analysis of thin rectangular flat plates carrying distributed load with the following boundary conditions.

Rectangular plate sides AD and BC, simply supported sides AB and DC cantilever supported sides and plate is loaded as in Figure 1 continuous load P_0 .

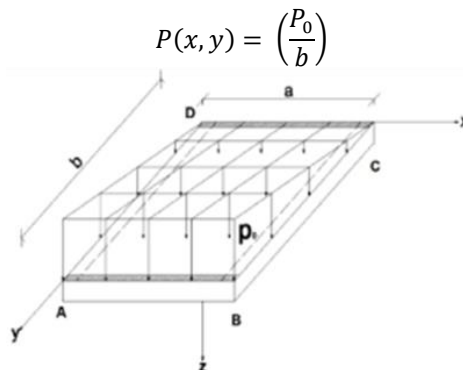


Figure 2. A rectangular plate with a continuous load.

In order to analyse plate FD methods used. First we should guess two function in x and y directions and after that boundary conditions must be suppose in this plate. For start imagine that plate is square and the dimension of both sides is a, in this case form function define as below:

$$X_m = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{a}\right)$$

$$Y_n = \sum_{n=1}^{\infty} \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

$$W(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \sin\left(\frac{n\pi y}{a}\right)$$

In case of consider the first set of statements

$$W(x,y) = w_0 \sin\left(\frac{\pi x}{a}\right) \left(\sin\left(\frac{\pi y}{a}\right)\right)^2$$

From Strain energy equation which stored in the plate

$$u = \frac{D}{2} \int_0^a \int_0^a \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\nu) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} \right\} dx dy$$

$$\frac{\partial^2 w}{\partial x^2} = -w_0 \frac{\pi^2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)^2}{a^2} \quad \frac{\partial^2 w}{\partial y^2} = w_0 \frac{2\pi^2 \cos\left(\frac{2\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right)}{a^2}$$

$$\frac{\partial^2 w}{\partial x \partial y} = w_0 \frac{\pi^2 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)}{a^2}$$

$$u = \frac{D}{2} \int_0^a \int_0^a \left\{ w_0^2 \left(-\frac{\pi^2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)^2}{a^2} + \frac{2\pi^2 \cos\left(\frac{2\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right)}{a^2} \right) + 2(1-\nu)^2 \right. \\ \left. + 2(1-\nu) \left\{ \left(\frac{\pi^2 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi y}{a}\right)}{a^2} \right)^2 \right. \right. \\ \left. \left. - \left(-\frac{\pi^2 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)^2}{a^2} \right) \left(\frac{2\pi^2 \cos\left(\frac{2\pi y}{a}\right) \sin\left(\frac{\pi x}{a}\right)}{a^2} \right) \right\} \right\} dx dy$$

$$u = \frac{27D\pi^4}{32a^2} W_0^2$$

Work done by the external forces are calculated as follows

$$w = \int_0^a \int_0^a p(x,y) w(x,y) dx dy = \int_0^a \int_0^a \left\{ \left(\frac{p_0}{a} \right) y + w_0 \sin\left(\frac{\pi x}{a}\right) \left(\sin\left(\frac{\pi y}{a}\right) \right)^2 \right\} dx dy = \frac{a^2 w_0 p_0}{2\pi}$$

can be minimized the potential function of plate to obtain the maximum deflection

$$\Pi = U - W = \frac{27D\pi^4}{32a^2} W_0^2 - \frac{a^2 w_0 p_0}{2\pi}$$

$$\frac{\partial \Pi}{\partial w_0} = \frac{27D\pi^4}{16a^2} w_0 - \frac{p_0}{2\pi} = 0$$

$$w_0 = \frac{8a^4 p_0}{27D\pi^5}$$

Finally, The deformation of the plate is as follows:

$$W(x,y) = \left(\frac{8a^4 p_0}{27D\pi^5} \right) \sin\left(\frac{\pi x}{a}\right) \left(\sin\left(\frac{\pi y}{a}\right) \right)^2$$

With the following values for the dimensions, numerical and analytical results will be compare

poisson's ratio	0.3	Length (cm)	100
continuous load (kg/ m ²)	10000	Width (cm)	100
Yeild Strenght (strain)	2400	Thickness (cm)	1
Yeild Strenght	2400	modulus of elasticity (kg/ cm ²)	2100000
Safety Factor design	1.667	Support situation	Two sides simply supported and two sides cantilever supported

The maximum deflection of plate In the middle of the plate from the last equation is as follows:

$$W(50,50) = \left(\frac{8(100)^4(1)}{27 \left(\frac{2.1 \times 10^6}{12(1 - 0.3^2)} \right) \pi^5} \right) \sin\left(\frac{\pi 50}{100}\right) \left(\sin\left(\frac{\pi 50}{100}\right) \right)^2 = 0.5035 \text{ cm}$$

Table 1. numerical and analytical deflections with different mesh size

Mesh size	numerical deflection	analytical deflection	Error percent
10	0.527663	0.503478	- 4.803586254
12	0.519066	0.503478	- 3.09606378
14	0.513746	0.503478	- 2.039413837
16	0.511844	0.503478	- 1.661641621
20	0.509534	0.503478	- 1.202833093
24	0.507886	0.503478	- 0.875509953
26	0.507231	0.503478	- 0.745414894
28	0.506659	0.503478	- 0.632003782
30	0.506158	0.503478	- 0.532297341
32	0.505713	0.503478	- 0.443912147
34	0.505317	0.503478	- 0.365259257
36	0.504962	0.503478	- 0.294749721
40	0.504351	0.503478	- 0.173393872
42	0.504056	0.503478	- 0.120759993
44	0.503844	0.503478	- 0.072694338
46	0.503734	0.503478	- 0.050846313
48	0.503741	0.503478	- 0.052236642
50	0.503729	0.503478	- 0.049853221
54	0.503669	0.503478	- 0.037737498
58	0.503572	0.503478	- 0.018670131
60	0.503516	0.503478	- 0.0075475
62	0.503453	0.503478	- 0.004369605
64	0.503394	0.503478	- 0.016683946
66	0.50333	0.503478	- 0.029395525
70	0.5032	0.503478	- 0.055215918
78	0.503046	0.503478	- 0.085803153
80	0.503056	0.503478	- 0.083816969
90	0.503024	0.503478	- 0.090172758
100	0.502912	0.503478	- 0.11241802
120	0.502851	0.503478	- 0.124533743
140	0.502757	0.503478	- 0.14124769
160	0.502746	0.503478	- 0.145984532
180	0.502728	0.503478	- 0.148963808
200	0.502696	0.503478	- 0.155319597
250	0.502677	0.503478	- 0.159093347

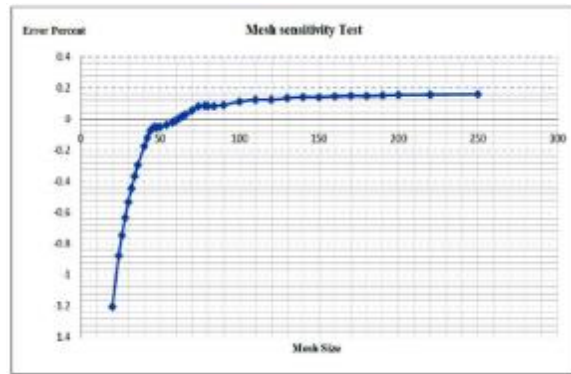


Figure 3. Mesh sensitivity

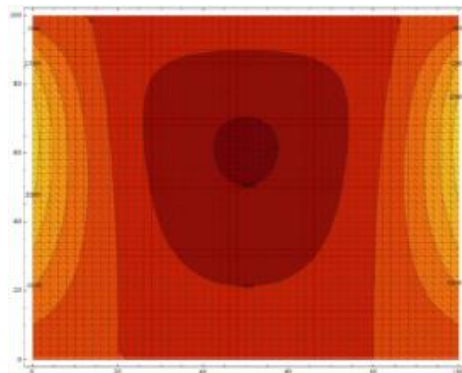


Figure 4. X- Axis normal stress counters (σ_x)

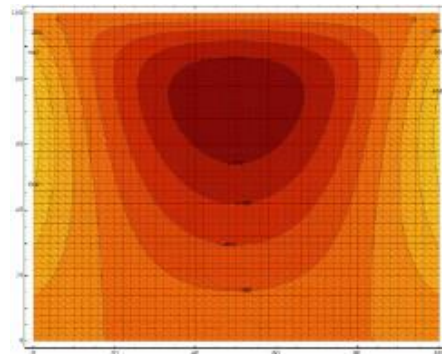


Figure 5. y- Axis normal stress counters (σ_y)

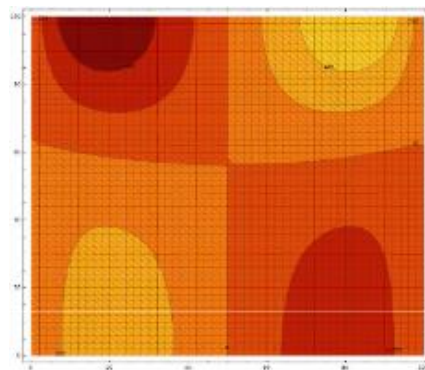


Figure 5. xy- Axis shear stress counters (τ_{xy})

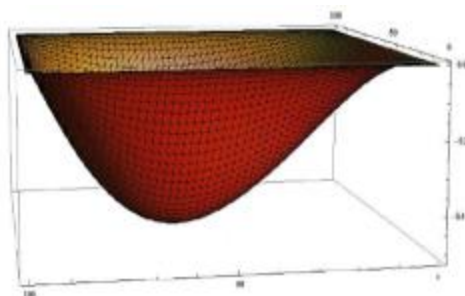


Figure 6. Stress function graph

CONCLUSION

Given the difficulty and long analytical solution of the plate equations in some specific issues, using numerical methods is always a good practice, provided that the authenticity and accuracy of these methods can be evaluated and selecting optimized steps for iteration, both save time during the analysis of problems and get appropriate and acceptable accuracy. Finite difference method as one of the existing numerical methods is relatively strong method for numerical solution of the plate equations with different loading and support solutions conditions. As observed, increasing repeat steps in finite difference method does not result in increased attention to the problem and may also act in the opposite way.

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